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Convergence Rates of Padé and Padé-Type Approximants

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A comparison is made between Padé and Padé-type approximants. Let Q_n be the nth orthonormal polynomial with respect to a positive measure μ with compact support in $\mathbf{C}$. We show that for functions of the form

$$f(z)=\int \frac{w(t)}{z-t}\,d\mu(t),$$

where w is an analytic function on the support of μ, Padé-type approximants with denominator Q_n give a successful and, in general, better approximation procedure than Padé approximation.

1. INTRODUCTION

In any approximation problem there are three main questions.

Q1. For how large class of functions may the approximation scheme be applied?

Q2. How easily and within what computational accuracy may the approximants be constructed?

Q3. Which is the theoretical accuracy (rate of convergence)?

We want to compare Padé and Padé-type approximants in a discussion on these three questions. First, however, we define our approximants. We

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shall use approximants interpolating at infinity to a function of one complex variable which is analytic at infinity.

1.1. *Definitions*

Let f be analytic at infinity. The *diagonal Padé approximant* (PA) at infinity *of order n of f* is the unique rational function p_n/q_n such that $p_n(z)$ and $q_n(z)$ are polynomials in z, $q_n(z) \not\equiv 0$, the degree of q_n, $\deg q_n$, is at most n, and

$$q_n(z)\, f(z) - p_n(z) = \mathcal{O}(z^{-n-1}), \qquad \text{as} \quad z \to \infty,$$

where the right-hand side denotes a power series in z^{-1} with lowest order term of degree $n+1$ or higher.

The definition of PAs means that we determine p_n and q_n by interpolating at infinity to f of as high degree as possible. If instead we fix some or all of the poles of the approximant in advance we get Padé-type approximants instead of PAs. In this paper we shall consider only the case when *all* poles are preassigned. More precisely, let $Q_n \not\equiv 0$, be a given polynomial in z of degree at most n. The *Padé-type approximant* (PTA) at infinity *of order n of f with preassigned denominator* Q_n (i.e. with preassigned poles at the zeros of Q_n), is the unique rational function P_n/Q_n such that P_n is a polynomial in z and

$$Q_n(z)\, f(z) - P_n(z) = \mathcal{O}(z^{-1}), \qquad \text{as} \quad z \to \infty.$$

The definitions mean that the Padé numerator p_n and the Padé-type numerator P_n are the polynomial parts of the Laurent series at infinity of $q_n f$ and $Q_n f$, respectively. Observe that in general p_n and P_n are polynomials of degree n. In the theorems in this paper the function f is zero at infinity which means that p_n and P_n have degree at most $n-1$. We shall indicate this by using the notation p_{n-1} and P_{n-1}.

1.2. *Discussion*

Q1: The advantage of PTAs compared to PAs is evident. One of the few general results where diagonal PAs guarantee convergence is Markov's theorem for functions of the type

$$f(z) = \int_\Delta \frac{d\mu(t)}{z-t}, \qquad \Delta \subset \mathbf{R}, \qquad \Delta \text{ compact interval}, \tag{1}$$

where μ is a positive measure on Δ. But PTAs guarantee convergence for the whole class of functions analytic in $\overline{\mathbf{C}} \setminus \Delta$. In fact, if in the definition (1) of f we replace Δ by some curve in $\mathbf{C}$ or even by an arbitrary regular compact set $K \subset \mathbf{C}$, and we allow μ to be a complex measure, we again have a convergence result for PTAs but not for PAs. In particular, f may be a Cauchy type integral. Moreover, we have convergence for the whole class of functions analytic in the unbounded component of $\overline{\mathbb{C}}\setminus K$, and even for the class of meromorphic functions in $\overline{\mathbb{C}}\setminus K$ (see [8], p. 200, Theorem III b for analytic functions and [3], Theorem 3 for the general case).

Q2: We have already remarked that both for PAs and PTAs the numerator of an approximant of a function f is the polynomial part of the Laurent series at infinity of the product of f and the denominator of the approximant. Hence, the calculation of the numerator of an approximant is a quite easy and stable procedure.

The core of the problem of constructing the nth diagonal PA is the numerical solution of a system of n linear equations in order to find the $n+1$ coefficients of the denominator q_n. If the determinant of this system is close to zero, this numerical solution is an ill-posed problem. Note also that if the determinant is zero we may get a wrong result after rounding off (see [4], p. 61). On the other hand, if we use PTAs, for example for the case of functions analytic in $\overline{\mathbb{C}}\setminus\Delta$, $\Delta=[-1, 1]$, we can use, in particular, any classical system of orthogonal polynomials on Δ (Legendre, Chebyshev, etc.) (see [8], p. 200, Theorem III b or [3], Theorem 3). We have these polynomials ready beforehand and so it takes no time or effort to calculate the denominator of the PTAs.

Q3: The principal question in any approximation problem is how to use the information available in the optimal way. As pointed out in [4], p. 63, the actual calculation of the PAs of f is usually the least time consuming part of a complete calculation. Rather, it is the computation of the power series coefficients of f to high accuracy which is expensive in computer time.

Ordinary PAs p_{n-1}/q_n of the function (1) give the following rate of convergence

$$\limsup_{n\to\infty}\left|f(z)-\frac{p_{n-1}(z)}{q_n(z)}\right|^{1/n}\leqslant e^{-2g_\Omega(z)} \tag{2}$$

locally uniformly (on compact subsets) in $\Omega=\overline{\mathbb{C}}\setminus\Delta$, where $g_\Omega(z)$ is the Green function with pole at infinity of Ω. If we consider PTAs P_{n-1}/Q_n of the same function (1) they give the rate (see [8], p. 200 or [3], Theorem 3)

$$\limsup_{n\to\infty}\left|f(z)-\frac{P_{n-1}(z)}{Q_n(z)}\right|^{1/n}\leqslant e^{-g_\Omega(z)} \tag{3}$$

locally uniformly in $\Omega=\overline{\mathbb{C}}\setminus\Delta$.

Note that the right-hand side of (2) differs from the right-hand side of (3) by a factor 2 in the exponent, and that it may seem that ordinary PAs give twice as high rate of convergence as PTAs. However, the basic fact here is that to construct the diagonal PA of order n of f we need to know the first $2n$ power series coefficients of f but to construct the PTA of order n we need just the n first coefficients of f. Consequently, if we have information about the first $2n$ coefficients of f, we can either construct the ordinary diagonal PA of order n or the PTA of order $2n$. They provide, as (2) and (3) show, the same rate of convergence. Note also that, as we have discussed above, calculating PTAs on a computer is much easier than calculating PAs.

But this is not the end of the story on Q3. The surprising fact is that for quite general classes of functions PTAs give better rate of convergence than PAs. This is the consequence of our main results, Theorem 1 and 2 in Section 2 (see also Remark 7). In Theorem 3 in Section 2 we make a further comparison between PAs and PTAs.

This paper is the third in a series of papers by the authors devoted to showing that PTAs are useful complements and substitutes to PAs. The first two papers are [2] and [3] and further papers are under preparation.

2. RESULTS

2.1. PTAs provide better rate of convergence in particular for the following class of Markov functions

$$f(z)=\int_{\Delta}\frac{w(t)}{z-t}\,dt,\quad \Delta=[-1,1], \tag{4}$$

where the weight w is an entire function (see Theorem 1 below). In this case as denominators of the PTAs P_{n-1}/Q_n we use the orthonormal polynomials Q_n with respect to Lebesgue measure dt on Δ, the so called Legendre polynomials. Observe that w may change sign on Δ. The typical case considered in rational approximation is when w is a polynomial.

THEOREM 1. *Let Q_n be the orthonormal Legendre polynomial of degree n and let P_{n-1}/Q_n be the PTA of order n of the function* (4) *with preassigned denominator Q_n. Suppose that w in* (4) *is an entire function. Then*

$$\limsup_{n\to\infty}\left|f(z)-\frac{P_{n-1}(z)}{Q_n(z)}\right|^{1/n}\leqslant e^{-2g_{\Omega}(z)} \tag{5}$$

locally uniformly in $\Omega=\overline{\mathbb{C}}\setminus\Delta$.

Remark 1. If we compare (5) with (2) we note that PTA and PA of order n give the same rate of convergence. However, as we have mentioned above to construct PTA and PA we need n and $2n$ coefficients, respectively.

Theorem 1 remains valid if in (4) we replace $w(t)\,dt$ by $w(t)\,d\mu(t)$, where μ is a positive measure with compact support in Δ, if we let Q_n be orthonormal polynomials with respect to μ. In this form the theorem means that the polynomials Q_n serve not only the Markov function (1) generated by $d\mu(t)$ but also with the same success the whole class of Markov functions generated by real measures $w(t)d\mu(t)$ where w is an entire function. If, in particular, μ is any classical measure we obtain classical orthonormal polynomials Q_n as denominators of the corresponding PTAs. Furthermore, Theorem 1 remains valid also if we replace the interval Δ in (4) by an arbitrary compact set in $\mathbf{C}$. More precisely, we shall prove the following theorem which contains Theorem 1 as a special case

THEOREM 2. *Let*

$$f(z)=\int \frac{w(t)}{z-t}\,d\mu(t) \tag{6}$$

where w is an entire function and μ a finite positive Borel measure on $\mathbf{C}$ with compact support, $\operatorname{supp}\mu$. *Let Q_n be the orthonormal polynomial of degree n with respect to μ, and P_{n-1}/Q_n the PTA of order n of f with preassigned denominator Q_n. Finally, let Ω be the unbounded component of* $\overline{\mathbf{C}}\backslash\operatorname{supp}\mu$. *Then* (5) *holds locally uniformly in* $\overline{\mathbf{C}}\backslash Co(\operatorname{supp}\mu)$ *where* $Co(\operatorname{supp}\mu)$ *denotes the convex hull of* $\operatorname{supp}\mu$. *Furthermore,*

$$\liminf_{n\to\infty}\left|f(z)-\frac{P_{n-1}(z)}{Q_n(z)}\right|^{1/n}\leqslant e^{-2g_\Omega(z)} \qquad \text{in } \Omega. \tag{7}$$

In (5) and (7) $g_\Omega(z)$ denotes the generalized Green function if $\operatorname{supp}\mu$ has positive logarithmic capacity and Ω is an irregular domain. If $\operatorname{supp}\mu$ has capacity zero, $g_\Omega(z)$ is infinite everywhere in Ω.

There is a version of Theorem 2 also when w is not an entire function; see Remark 7 in Section 3.

Remark 2. It is not possible to replace limes inferior by limes superior in (7). This depends on the fact that Q_n may have zeros in $\Omega\cap Co(\operatorname{supp}\mu)$ if that set is non-empty (see for instance [7], p. 31).

Remark 3. For functions of the form (6) we can no longer expect local uniform convergence in Ω for the diagonal PAs. In fact, consider the function (6) where $\operatorname{supp}\mu=\Delta=[-1,1]$, $d\mu(t)=dt/\sqrt{1-t^2}$, and

$w(t) = (t - a_1)(t - a_2)$ with $a_j = \sin \pi\alpha_j$, $j = 1, 2$, and $1, \alpha_1, \alpha_2$ rationally independent real numbers. In this case w changes sign in $(-1, 1)$ and Stahl [6] has proved that the sequence of diagonal PAs has poles asymptotically dense in $\mathbf{C}$. Hence, the diagonal PAs do not converge locally uniformly anywhere in Ω.

2.2. We now return to a function f given by (4) with a fixed weight w. Let P_{n-1}/Q_n be the PTA of order n of f with preassigned denominator Q_n where Q_n is the orthonormal Legendre polynomial of degree n. We introduce the error

$$R_n(z) = f(z) - \frac{P_{n-1}(z)}{Q_n(z)}. \tag{8}$$

By $r_n(z)$ we denote the error (8) for the special case of the function (4) when $w(t) \equiv 1$. In this case the PTAs introduced coincide with ordinary PAs. The following theorem asserts that the errors $R_n(z)$ and $r_n(z)$ are of the same order.

THEOREM 3. *If $R_n(z)$ and $r_n(z)$ are the errors introduced above and w is an entire function, then*

$$\limsup_{n \to \infty} |R_n(z) - w(z)\, r_n(z)|^{1/n} = 0 \tag{9}$$

locally uniformly in $\overline{\mathbf{C}} \setminus \Delta$.

Remark 4. Theorem 3 means that the difference $R_n(z) - w(z)\, r_n(z)$ tends to zero faster than any geometric progression as n tends to infinity. But note that $r_n(z)$ is the error of the ordinary diagonal PA of order n of the simplest function

$$f(z) = \int_\Delta \frac{dt}{z - t}, \quad \Delta = [-1, 1], \tag{10}$$

with the Lebesgue measure dt. It is natural to expect that in general the simplest function (10) is approximated better by PAs than the function (4) with more general weights w. Consequently, if what we expect is true, due to (9) the function (4) with a general entire function w must be approximated better by PTAs of order n than by ordinary PAs of the same order. The analogous result may be formulated aslo for functions of the form (6).

3. PROOFS

Theorem 1 is a special case of Theorem 2.

Proof of Theorem 2. We first prove that (5) holds in $\bar{\mathbf{C}}\setminus Co(\operatorname{supp}\mu)$. By the definition of f and of the PTAs we obtain

$$Q_n(z)\, f(z) = Q_n(z) \int \frac{w(t)}{z-t}\, d\mu(t) = \int \frac{(Q_n(z) - Q_n(t)) + Q_n(t)}{z-t}\, w(t)\, d\mu(t)$$

$$= P_{n-1}(z) + \int \frac{Q_n(t)\, w(t)}{z-t}\, d\mu(t)$$

since the polynomial part of the Laurent expansion at infinity of the last integral is zero. Hence,

$$f(z) - \frac{P_{n-1}(z)}{Q_n(z)} = \frac{1}{Q_n(z)} \int \frac{Q_n(t) w(t)}{z-t}\, d\mu(t). \tag{11}$$

In our estimate of the right-hand side of (11) we shall use the following inequality on $Q_n(z)$ (see [7], p. 4)

$$\liminf_{n\to\infty} |Q_n(z)|^{1/n} \geqslant e^{g_\Omega(z)} \tag{12}$$

locally uniformly in $\mathbf{C}\setminus Co(\operatorname{supp}\mu)$. Since Q_n is the nth orthogonal polynomial with respect to μ, the other factor in the right-hand side of (11) can be written in the following form, where $\pi_{n-1}(t)$ is any polynomial in t of degree at most $n-1$,

$$\int \frac{Q_n(t) w(t)}{z-t}\, d\mu(t) = \int Q_n(t) \left[\frac{w(t)}{z-t} - \pi_{n-1}(t) \right] d\mu(t). \tag{13}$$

We choose $\pi_{n-1}(t)$, depending on z, so that it approximates $w(t)/(z-t)$ as good as possible on $\operatorname{supp}\mu$ in the supremum norm. Then we use the following lemma.

LEMMA 1. *Let w be an entire function and K a compact subset of $\mathbf{C}$. Introduce the approximation error*

$$E_n(z) = \inf_{\pi_n} \sup_{t\in K} \left| \frac{w(t)}{z-t} - \pi_n(t) \right|,$$

where the infimum is taken over all polynomials π_n *of degree at most n. Then*

$$\limsup_{n\to\infty} E_n(z)^{1/n} \leqslant e^{-g_\Omega(z)} \tag{14}$$

locally uniformly in the unbounded component Ω *of* $\overline{\mathbf{C}}\backslash K$.

Before we comment on the proof of the lemma we finish the proof of Theorem 2. From (13) we get by using the notation of the lemma with $K = \operatorname{supp}\mu$ and the fact that Q_n is the orthonormal polynomial of degree n with respect to μ,

$$\left|\int \frac{Q_n(t)\, w(t)}{z-t}\, d\mu(t)\right| \leqslant E_{n-1}(z) \int |Q_n(t)|\, d\mu(t)$$

$$\leqslant E_{n-1}(z) \int (|Q_n(t)|^2 + 1)\, d\mu(t) = E_{n-1}(z) \cdot (1 + \mu(\mathbf{C})).$$

If we use the lemma on this estimate and combine this with (11) and (12), we obtain (5).

The proof of (7) is the same as the proof of (5) except that (12) is replaced by (see [1], Theorem 1)

$$\limsup_{n\to\infty} |Q_n(z)|^{1/n} \geqslant e^{g_\Omega(z)} \qquad \text{everywhere in } \Omega. \tag{15}$$

Remark 5. There is a version of (15) (see [1], Theorem 1′) giving uniform convergence in a neighbourhood of an arbitrary point of Ω. That gives a corresponding version of (7) with uniform convergence.

Proof of Lemma 1. This lemma is well-known (see [8], p. 154 and 85 or [5], p. 74) except maybe the complication caused by the parameter z. Because of that we only sketch the proof and refer to [8] or [5] for details.

Step 1. First we consider the case when K has positive logarithmic capacity and is a regular set in the sense that Ω has a classical Green function $g_\Omega(z)$. We introduce the level curves

$$C_\rho = \{s \in \mathbf{C}\colon g_\Omega(s) = \log \rho\},\ \rho > 1.$$

Let $z_j^{(n)}$, $0 \leqslant j \leqslant n$, be the Fekete points on K and $\pi_n(\cdot) = \pi_n(\cdot, z)$ the interpolation polynomial of degree n interpolating at the Fekete points to the function $F(\cdot) = F(\cdot, z)$ defined by $F(t, z) = w(t)/(z-t)$. Introduce $\omega_n(t) = \prod_{j=0}^{n} (t - z_j^{(n)})$.

By Hermite's interpolation formula

$$F(t, z) - \pi_n(t, z) = \frac{1}{2\pi i} \int_{C_R} \frac{\omega_n(t)}{\omega_n(s)} \cdot \frac{F(s, z)}{s - t} \, ds,$$

for $t \in K$, $z \in \Omega$, where $1 < R < e^{g_\Omega(z)}$. Notice that for each fixed $z \in \Omega$, $F(\cdot, z)$ is analytic inside C_ρ with $\rho = \exp\{g_\Omega(z)\}$. Since the error $F(t, z) - \pi_n(t, z)$ depends on the parameter z in a simple way, we can use Hermite's formula essentially as in [8] or [5] to complete the proof.

Step 2. We now treat the case when K is an irregular set or a set of capacity zero. Fix $z \in \Omega$ and a compact set F such that $z \in F \subset \Omega$. We exhaust Ω by an increasing sequence S of open regular sets. If cap $K > 0$ we choose $\delta > 1$ and a set Ω_1 in this sequence S so that $\Omega_1 \supset F$ and $g_\Omega < g_{\Omega_1} + \log \delta$ on F. If cap $K = 0$ we choose $N > 0$ and Ω_1 in S so that $\Omega_1 \supset F$ and $g_{\Omega_1} > N$ on F. We then use Step 1 with K replaced by $\mathbf{C} \setminus \Omega_1$. By choosing δ close to 1 and N large, we obtain the lemma.

Remark 6. In the proof below of Theorem 3 we need a version of the lemma where $w(t)/(t - z)$ is replaced by $F(t, z) = (w(t) - w(z))/(t - z)$. In this case $F(\cdot, z)$ is an entire function and the right-hand side of (14) may be replaced by zero.

Remark 7. There is a version of the lemma and of Theorem 2 also when w is not an entire function. Suppose that w is analytic inside C_R, $R > 1$, and that $z \in \Omega$. Then (5) and (7) hold if z is inside C_R. Otherwise the right-hand side of (5) and (7) shall be replaced by $R^{-1} \cdot \exp\{-g_\Omega(z)\}$.

Proof of Theorem 3. Let f be given by (4), use (11) and consider

$$R_n(z) = f(z) - \frac{P_{n-1}(z)}{Q_n(z)} = \frac{1}{Q_n(z)} \int_\Delta \frac{Q_n(t)\, w(t)}{z - t} \, dt$$

$$= \frac{1}{Q_n(z)} \int_\Delta \frac{Q_n(t)(w(t) - w(z))}{z - t} \, dt + \frac{w(z)}{Q_n(z)} \int_\Delta \frac{Q_n(t)}{z - t} \, dt. \tag{16}$$

By the same arguments as in the proof of Theorem 2 combined with Remark 6 we get

$$\limsup_{n \to \infty} \left| \frac{1}{Q_n(z)} \int_\Delta \frac{Q_n(t)(w(t) - w(z))}{z - t} \right|^{1/n} = 0 \tag{17}$$

locally uniformly in $\overline{\mathbf{C}}\backslash\Delta$. Now we note that

$$r_n(z) = \frac{1}{Q_n(z)} \int_\Delta \frac{Q_n(t)}{z-t}\, dt. \tag{18}$$

Theorem 3 follows from (16), (17) and (18).

REFERENCES

1. A. Ambroladze, On exceptional sets of asymptotic relations for general orthogonal polynomials, *J. Approx. Theory* **82** (1995), 257–273.
2. A. Ambroladze and H. Wallin, Approximation by repeated Padé approximants, *J. Comp. Applied Math.* **62** (1995), 353–358.
3. A. Ambroladze and H. Wallin, Padé type approximants of Markov and meromorphic functions, *J. Approx. Theory*, to appear.
4. G. A. Baker, Jr., and P. R. Graves-Morris, Padé approximants, I, *in* "Encyclopedia of Math.," Vol. 13, Addison–Wesley, Reading, MA, 1981.
5. D. Gaier, "Lectures on Complex Approximation," translation of "Vorlesungen über Approximation in Komplexen," Birkhäuser-Verlag, Basel, 1980.
6. H. Stahl, On the divergence of certain Padé approximants and the behaviour of the associated orthogonal polynomials, *in* "Polynomes Orthogonaux et Applications, Proceedings, Bar-le-Duc, 1984" (C. Brezinsky *et al.*, Eds.), Springer Lecture Notes in Math., Vol. 1171, pp. 321–330, Springer-Verlag, New York/Berlin, 1985.
7. H. Stahl and V. Totik, General orthogonal polynomials, *in* "Encyclopedia of Math.," Cambridge Univ. Press, Cambridge, 1992.
8. J. L. Walsh, "Interpolation and Approximation by Rational Functions in the Complex Domain," Amer. Math. Soc. Colloq. Publ., 4th ed., Vol. 20, Amer. Math. Soc., Providence, RI, 1965.